

WORLD 2 / CHAPTER 1**FINANCIAL MATHEMATICS**

	Page
1. Methods of interest calculation, Yield curve and quotation	2
1.1 Methods to calculate interest	2
1.2 The yield curve	7
1.3 Interpolation	9
1.4 Quotation	10
2. Financial Arithmetics in the Money Market	11
2.1 Calculating simple interest	11
2.2 Average interest	12
2.3 Calculating compound interest (effective interest)	13
2.4 Calculating forward rates (for terms less than one year)	14
2.5 Calculating the future value (for terms less than one year)	15
2.6 Calculating the present value	17
2.7 Interest calculation with present value and future value	18
2.8 Real interest rate and nominal interest rate	19
2.9 Calculating the interest rate from a discount rate	20



FINANCIAL MATHEMATICS

1. Methods of interest calculation, yield curve and quotation

1.1 Methods to calculate interest

While calculating interests, the general question is how the interest for one period is determined. The interest calculation methods employed can vary, depending on national and product markets.

As a rule, interest can be calculated in the following manner:

$$I = C \cdot r \cdot \frac{D}{B}$$

I = amount of interest

C = capital amount

r = interest rate in decimals (i.e. 5 % = 0.05; 10.3 % = 0.103; etc.)

D = number of days of the term of interest

B = day basis for calculation (fixed number of days per year) (= day base)

There are three ways to determine the number of days (D).

a) **Actual:** Counting the actual numbers of days that elapse.

Term of interest: 1 March – 31 March → 30 days

Term of interest: 1 March – 1 April → 31 days

b) **30**: Each month counts as 30 days (remaining days in a month are subtracted).

Term of interest: 1 March – 31 March → 30 days

Term of interest: 1 March – 30 March → 29 days

Term of interest: 1 March – 1 April → 30 days

c) **30E**: Each month counts as 30 days (the 31st is treated as if it was the 30th; remaining days are subtracted).

Term of interest: 1 March – 31 March → 29 days

Term of interest: 1 March – 30 March → 29 days

Term of interest: 1 March – 1 April → 30 days

This method is used in the Euromarket as well as in some continental European markets.

There are three ways to determine the day basis (B).

a) **360**: Assuming that each year has 360 days.

Annual term: 1 March XY – 1 March XZ are 365 days → day base is 360 days

Annual term: 1 March XY – 3 March XZ*) are 367 days → day base is 360 days

**) e.g. with a weekend*

b) **365**: Assumption that each year has 365 days.

Just as with **360**, but generally → day base is 365 days

c) **Actual**:

- In the money market

The actual days per year are counted (leap year 366 days, "normal" year 365 days).

If a deal runs over two years (one of them being a leap year), the interest calculation is divided into two parts.

Example

2 January 1995 – 2 January 1996

interest calculation Actual/Actual

$$x \cdot \frac{364}{365} + x \cdot \frac{1}{366}$$

- In the capital market
A year is counted with actual days of the term of interest (multiplied by the number of terms of interest)

Example

Bond with semi-annual interest payments:

1 March XY – 1 September XY are 183 days → day basis is 366 days
(183·2)

$$x \cdot \frac{183}{366}$$

Therefore, theoretically nine combinations of days (D) and basis (B) are possible but only 5 of them are practically used: Actual / 365; Actual / 360; 30 / 360; 30E / 360; and Actual / Actual.

Daily conventions vary from market to market. In the table below the conventions for money markets and capital markets are listed. In the capital markets, however, these conventions may differ in their specifications regarding the international and domestic market and regarding different financial instruments.

Therefore, please clarify these conditions before you trade!

<i>Money market</i>		<i>Money market</i>		<i>Capital Market</i>	
Australia	Actual / 360	Norway	Actual / 360	Euro	Actual / Actual
Euro	Actual / 360	Poland	Actual / 365	Great-Britain	Actual / Actual
Greece	Actual / 365	Sweden	Actual / 360	Japan	30 / 360 or Actual / Actual
Great-Britain	Actual / 365	Switzerland	Actual / 360	Sweden	30 / 360 or 30E / 360
Hong Kong	Actual / 365	Czech Republic	Actual / 360	Switzerland	30 / 360 or 30E / 360
Japan	Actual / 360	USA	Actual / 360	USA	30 / 360 or Actual / Actual

Example

Semi-annual bond, principal 10,000 at an interest rate of 7.00 % p.a., last coupon on 1 May, next coupon on 1 November (number of days: 184). On 31 May, the following interest is due:

<i>Calculation method</i>	<i>Days of term/Days per year</i>	<i>Calculation of interest</i>
Actual / 365	30 / 365	$10,000 \cdot 0.075 \cdot \frac{30}{365} = 61.64$
Actual / 360	30 / 360	$10,000 \cdot 0.075 \cdot \frac{30}{360} = 62.50$
30 / 360	30 / 360	$10,000 \cdot 0.075 \cdot \frac{30}{360} = 62.50$
30E / 360	29 / 360	$10,000 \cdot 0.075 \cdot \frac{29}{360} = 60.42$
Actual / Actual	30 / 368	$10,000 \cdot 0.075 \cdot \frac{30}{368} = 61.14$

1.2 The yield curve

The interest rate structure or yield curve plots on the x-axis the term of interest and at y-axis the corresponding interest rate. It refers to a certain date and shows how much interest is paid for different terms. Interest and yield curves can be determined for a number of financial instruments. Common examples are the yield curve in the money market (e.g. for LIBOR), or the yield curve for government bonds (long-term).

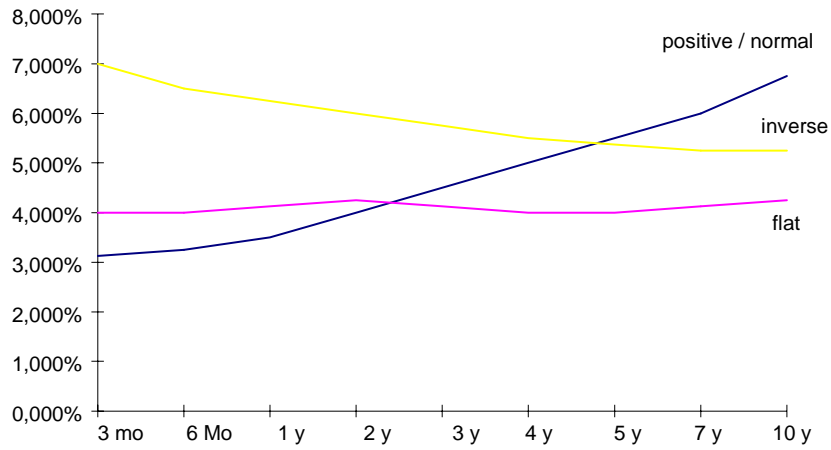
A long-term yield curve can be derived by determining and joining yields of bonds that have different times to maturity, typically 2, 3, 4, 5, 7, 10, 15, 20, or 30 years. Each bond's yield is plotted in a graph, and the dots are joined.

There are three different types of yield curves:

- a) **Steep yield curve:** ("normal") short-term interest rates are lower than long-term interest rates.
- b) **Flat yield curve:** interest rates for different terms are the same.
- c) **Inverse yield curve:** short-term interest rates are higher than long-term interest rates.

There are several theories on the shape of a yield curve. One of them is the so-called interest expectation theory. Therefore, the yield curve's represents the expectations of market participants regarding future yields. If the market expects rising interest rates, the yield curve will slope upwards, because market participants will anticipate higher future rates for long-term investments. Furthermore, it has to taken into account that a long-term business bears higher risks than a short-term business. Therefore, even a flat yield curve, where no interest rate change is expected, will slightly slope upwards. This constellation is called "normal interest curve".

Certainly, there are several other factors that have an influence on the yield curve, e.g. interventions of the central bank, preferences of liquidity among market participants, etc.



graph 1: yield curve

1.3 Interpolation

Since there is not always a benchmark at hand, interest rates must sometimes be estimated.

In the following, we want to show the most simple method to calculate interest rates for unusual terms: straight-line interpolation.

$$r = r_s + \left[\frac{r_l - r_s}{D_l - D_s} \right] \cdot (D - D_s)$$

- r = interest rate in decimals
- D = number of days of the term of interest
- r_s = interest rate in decimals, short-term
- D_s = number of days, short-term
- r_l = interest rate in decimals, long-term
- D_l = number of days, long-term

Example

We calculate the interest rate for a deposit for 1½ months (46 days) on the basis of the following interest rates:

- 1 month 3½ % (31 days)
- 3 months 3¾ % (92 days)

$$r = 0.035 + \left[\frac{0.0375 - 0.035}{92 - 31} \right] \cdot (46 - 31) = 3,56147 \%$$

The method of straight-line interpolation has the advantage that it is very easy to apply. To give a complete overview, we would like to point out, that there exist also methods which take the non-linearity of interest rates into account. These are:

- logarithmic interpolation
- cubic splines

1.4 Quotation

Interest rates are often quoted in basis points that lie either below or above a certain benchmark rate. A bank could lend money at LIBOR – 3 basis points; or a Dollar bond is issued at T + 50: this would mean 50 basis points above the yield of a comparable US Treasury bill. One basis point is equal to $\frac{1}{100}$ th of 1 %, i.e. 0.01 %.

The quotation of interest rates in the money market is done on a p.a. basis, where the interest is paid at the end of the term. The interest that is thereby paid is called simple interest (e.g. CHF $2\frac{1}{4} - 2\frac{3}{8}$ % for 6 months).

In the money market, most transactions have terms of not more than one year. Where the term exceeds 12 months, interest is paid first after one year and then at maturity. For example, interest payments on an 18-month deposit are due after 12 months for the first time, and finally after 18 months, i.e. at the end of the 6-month period of the second year. Simple interest for a definite period is calculated on the basis of the principal.

Note: There are different methods to calculate interest for individual instruments.

**Example**

What is the value of 1 BP of a USD 1,000,000 deposit yielding 5 % ?

Since USD is calculated on ACT /360 basis but the interest is quoted on the basis of a 360-day year, the value of 1 BP is

$$1,000,000 \cdot 0.05 \cdot 0.0001 \cdot \frac{365}{360} = 5.0694$$

2. Financial arithmetics in the money market

2.1 Calculating simple interest

The formula for calculating simple interest (single payment of interest and a term of less than one year) is:

$$I = C \cdot r \cdot \frac{D}{B}$$

I = amount of interest

C = capital amount

r = interest rate in decimals (i.e. 5 % = 0.05; 10.3 % = 0.103; etc.)

D = number of days of interest period

B = day basis of calculation (fixed number of days per year)

Example

Bank A gives a 1-month deposit of Euro 5 Mio at 3 %. Start date of this credit is 1 October and end date is 1 November. The actual number of days for this period is 31. Basis of term calculation is 360 days per year. The absolute interest of this credit is:

$$I = 5,000,000 \cdot 0.03 \cdot \frac{31}{360} = 12,916.67$$

On 1 November, the borrower will either

a) pay interest of Euro 12,961.67 and roll over the credit, or

b) settle the credit by paying back the principal plus interest: Euro 5,012,916.67

2.2 Average interest

If different interest rates apply over several interest periods while giving or taking money, the average interest rate may be calculated like this:

$$r_{AV} = \left[\left(r_1 \cdot \frac{D_1}{B} \right) + \left(r_2 \cdot \frac{D_2}{B} \right) + \left(r_3 \cdot \frac{D_3}{B} \right) + \dots + \left(r_n \cdot \frac{D_n}{B} \right) \right] \cdot \frac{B}{D_N}$$

r_{AV} = average interest rate

r_1, r_2, \dots, r_n = interest rate of the respective terms of interest, in decimals

B = day basis of calculation

D_N = number of days of the total term

D_1, D_2, \dots, D_n = number of days of the respective term

Example

You lent EUR at the following rates during the last year:

2 January - 2 April	(90 days) at 2.5 %
2 April - 2 July	(91 days) at 2.75 %
2 July - 2 October	(92 days) at 2.875 %
2 October - 2 January	(92 days) at 3 %

Calculate the average interest rate.

$$\left[\left(0.025 \cdot \frac{90}{360} \right) + \left(0.0275 \cdot \frac{91}{360} \right) + \left(0.02875 \cdot \frac{92}{360} \right) + \left(0.03 \cdot \frac{92}{360} \right) \right] \cdot \frac{360}{365}$$

= **2.78288 %**

2.3 Calculating compound interest (effective interest)

If an amount is lent or borrowed over several terms and the interest payments are not paid out at the end of each term, the amount that is the basis for the interest calculation is raised by the amount of accrued interest. This is commonly known as capitalisation or compound interest.

The general formula for the calculation of compound interest is:

$$ER = \left\{ \left[\left(1 + r_1 \cdot \frac{D_1}{B} \right) \cdot \left(1 + r_2 \cdot \frac{D_2}{B} \right) \cdot \left(1 + r_3 \cdot \frac{D_3}{B} \right) \cdot \dots \cdot \left(1 + r_n \cdot \frac{D_n}{B} \right) \right] - 1 \right\} \cdot \frac{B}{D_N}$$

ER = effective interest rate

D_1, \dots, D_n = number of days of the respective term

B = day basis of calculation

D_N = number of days of the total term

$r_1 \dots r_n$ = interest rate of the respective terms of interest, in decimals

Example

Last year you invested CHF at the rates given below. The subsequent investment is made up of the original amount and the accrued interest.

- 2 January - 2 April (90 days) at 2.50 %
- 2 April - 2 July (91 days) at 2.75 %
- 2 July - 2 October (92 days) at 2.875 %
- 2 October - 2 January (92 days) at 3.0%

What is the effective interest rate ?

$$\left\{ \left[\left(1 + 0.025 \cdot \frac{90}{360} \right) \cdot \left(1 + 0.0275 \cdot \frac{91}{360} \right) \cdot \left(1 + 0.02875 \cdot \frac{92}{360} \right) \cdot \left(1 + 0.03 \cdot \frac{92}{360} \right) \right] - 1 \right\} \cdot \frac{360}{365}$$

= 2.81240%

2.4 Calculating forward rates (for terms less than one year)

A forward-forward rate (or simply forward rate) is an interest rate for a future term of interest, e.g. an interest rate for a 6-month investment that will begin in 3 months. These forward rates can be derived from the interest rates prevailing in the market. By investing for 9 months and refinancing for 3 months the same effects can be achieved today.

The formula to calculate forward rates is as follows:

$$FR = \left\{ \frac{\left[1 + \left(r_l \cdot \frac{D_l}{B} \right) \right]}{\left[1 + \left(r_s \cdot \frac{D_s}{B} \right) \right]} - 1 \right\} \cdot \frac{B}{D_{l-s}}$$

- FR = forward interest rate
- r_l = interest rate in decimals, long-term
- r_s = interest rate in decimals, short-term
- D_l = number of days, long-term
- D_s = number of days, short-term
- B = day basis of calculation
- D_{l-s} = difference between short term and long term (in days)

Example

Forward rate calculation for GBP, starting in 3 months for a term of 3 months.

Interest rates GBP: 3 months = 7½ % (91 days)
 6 months = 7¾ % (183 days)

$$FR = \left\{ \frac{\left[1 + \left(0.0775 \cdot \frac{183}{365} \right) \right]}{\left[1 + \left(0.075 \cdot \frac{91}{365} \right) \right]} - 1 \right\} \cdot \frac{365}{92} = 7.85049\%$$

2.5 Calculating the future value (for terms less than one year)

Starting with the present value today (principal), the future value can be determined. The amount of money that is due at the end of the loan's or deposit's term is made up of the original amount of capital plus the interest. This amount is called the future value.

For example, if, today, you invest GBP 100 for 1 year at 4 % p.a., you receive an amount of GBP 104 (Actual / 365) at the end of the year.

The present value of this investment is GBP 100.

The future value of this investment is GBP 104.

Formulae for the simple calculation of the future value:

$$\boxed{FV = PV \cdot \left(1 + \left(r \cdot \frac{D}{B}\right)\right)} \quad \text{or} \quad \boxed{FV = PV + \left(PV \cdot r \cdot \frac{D}{B}\right)}$$

FV = future value of the investment

PV = present value

r = interest rate in decimals

D = number of days of the term of interest

B = day basis of calculation

In other words, the future value of an investment is the amount of capital (PV) plus accrued interest.

Example

You take a deposit of EUR 1 Mio at a rate of 6 % p.a. for 92 days (Actual / 360). At the end of its term, the value of the deposit will be:

$$FV = 1,000,000 \cdot \left(1 + \left(0.06 \cdot \frac{92}{360} \right) \right) = \text{EUR } 1,015,333.33$$

Note: When calculating future values that include compound interest, one has to first compute the effective interest rate from the nominal interest rate. Only after this has been done, can one use the above formula.

2.6 Calculating the present value

The current value of a future cash flow is called present value. The present value is calculated by discounting the future value. Many markets, e.g. that for US T-bills, conventionally use discount rates.

$$PV = \frac{FV}{1 + \left(r \cdot \frac{D}{B} \right)}$$

- PV = present value (capital)
- FV = future value
- r = interest rate in decimals
- D = number of days
- B = day basis of calculation

Example

We know that the current yield of a US treasury bill is 5.50 %. The future value is 1,000,000 in 2 months (61 days). Calculate the present value of the T bill.

$$PV = \frac{1,000,000}{1 + \left(0.055 \cdot \frac{61}{360} \right)} = 990.766,61$$

2.7 Interest calculation with present value and future value

If we know the future value, the present value as well as the term of an investment and if there are no cash flows during the term, we can calculate the current yield (current market interest rate).

$$r = \frac{(FV - PV)}{PV} \cdot \frac{B}{D}$$

- FV = future value
- PV = present value
- B = day basis of calculation
- D = number of days

Example

A US treasury bill has a time to maturity of 82 days.

Present value: = 987,627

Future value: = 1,000,000

$$r = \frac{(1,000,000 - 987,627)}{987,627} \cdot \frac{360}{82} = 5.5001 \%$$

Note: You cannot use this formula if the investment has several cash flows.

2.8 Real interest rate and nominal interest rate

Time has an influence on the value of money. Normally, EUR 100 today, are worth more than EUR 100 in one year. There are two main reasons for this: First, you can profitably invest the money. At an assumed interest rate for riskless investments of 5 % p.a., today's EUR 100 will be equal to EUR 105 in one year. Secondly, inflation has to be taken into account. Therefore, money that is not invested but kept in cash, is worth more today than in one year. If the inflation rate is at 2 % p.a., you would have to pay in one year EUR 102 for a basket of commodities that is worth EUR 100 today.

The **nominal interest rate** is equal to the absolute interest rate paid on the capital (market interest rate).

The **real interest rate** shows how the purchasing power of the capital is changing. It is equal to the nominal interest rate minus the inflation rate.

$$NR = [(1 + RR) \cdot (1 + I)] - 1$$

$$RR = \left(\frac{1 + NR}{1 + I} \right) - 1$$

NR = nominal interest rate, in decimals

RR = real interest rate, in decimals

I = inflation rate, in decimals

Example

The current interest rate of a deposit is 7 % p.a. while the inflation rate is 4 % per year. The real interest rate can be calculated as follows:

$$RR = \left(\frac{1 + 0.07}{1 + 0.04} \right) - 1 = 2.88462 \%$$

2.9 Calculating the interest rate from a discount rate

Some financial instruments (e.g. US Treasury bills) are quoted on a discount rate basis, i.e. the interest is calculated on the basis of the future value and not on the invested capital.

$$r_d = \frac{(FV - PV)}{FV} \cdot \frac{B}{D}$$

FV = future value

PV = present value

r_d = discount rate, in decimals

B = day basis of calculation

D = number of days

Example

A 1 mio US Treasury bill (92 days) is offered at a discount rate of 5 %.
Considering the T-bill convention, you can calculate the price as follows:

$$\text{PRICE} = 1,000,000 - \left(1,000,000 \cdot 0.05 \cdot \frac{92}{360} \right) = 987,222.22$$

The following formula allows us to convert the discount rate into an p.a. interest rate :

$$r = \frac{r_d}{1 - \left(r_d \cdot \frac{D}{B} \right)}$$

- r = interest rate p.a.
r_d = discount rate, in decimals
D = number of days
B = day basis of calculation

Example

The discount rate of a US T-bill is 5 %, its term is 92 days.
The comparable p.a. rate of interest is:

$$r = \frac{0.05}{1 - \left(0.05 \cdot \frac{92}{360} \right)} = 5.06472 \%$$